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Coulomb and quantum oscillator problems in conical spaces with arbitrary dimensions

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Abstract

The Schrödinger equations for the Coulomb and the harmonic oscillator potentials are solved in the cosmic string conical spacetime. The spherical harmonics with angular deficit are introduced. The algebraic construction of the harmonic oscillator eigenfunction is performed through the introduction of non-local ladder operators. By exploring the hidden symmetry of the two-dimensional harmonic oscillator the eigenvalues for the angular momentum operators in three dimensions are reproduced. A generalization for N dimensions is performed for both Coulomb and harmonic oscillator problems in angular deficit spacetimes. The connection among the states and energies of both problems in these topologically non-trivial spacetimes is thus established.

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1. Introduction

There has been a growing interest in spacetimes with non-trivial topology and how this can affect some aspects in classical or quantum cosmological models as well as in quantum mechanics. This non-trivial topology is generated by topological defects such as monopoles, strings, domain walls and branes. Their formation is associated with phase transitions in the early universe where the vacuum is degenerate [1]. Nevertheless, stable domain walls and monopoles are disastrous for cosmological models [2]. However, strings cause no harm and can be a good candidate to produce several phenomena observed in the last decades such as gravitational lenses [3, 4], particle production [5] and microwave sky anisotropy [6]. The most interesting topic for our study is the metric structure of the cosmic string spacetime. The metric leads to a conic spacetime. Its locally flat geometry affects non-relativistic systems only through the non-trivial topology. Thus, a non-relativistic particle placed in the surroundings of a straight, infinite and static string will not suffer attraction by the cosmic string gravitational field [3].

For a cosmic string spacetime the metric tensor in cylindrical coordinates (ρ, z, ϕ) is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\rho^2)$ where $0 \leqslant \rho \leqslant \infty$, $-\infty \leqslant z \leqslant \infty$, $-\pi\alpha \leqslant \phi \leqslant \pi\alpha$ and $\alpha = 1 - 4G\mu$, with μ being the linear density of the cosmic string. The Laplace–Beltrami operator in these coordinates

$$\Box\psi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\psi \tag{1}$$

takes the same form as in the flat spacetime. When one takes the non-relativistic limit of a system with the dynamics described by a covariant quantum equation such as the Dirac or Klein–Gordon equations one will be led to the usual Schrödinger equation but with non-trivial boundary conditions

$$\Phi(\phi_0) = \Phi(\phi_0 + 2\pi\alpha) \tag{2}$$

imposed by the global structure of space. Equation (2) expresses the main physical effect of the cosmic string in spacetime introducing an unusual periodicity in the azimuthal variable. In spite of being locally flat the spacetime acquires a global topological defect giving the conic character to the space. The cosmic deficit angle $\delta = 2\pi(1-\alpha) = 8\pi G\mu$ connects the topology of space with the matter content expressed by μ and the conical aspect affects any quantum wave solution that significantly encircles the string.

It has been noted that the cosmic string spacetime affects the quantum solutions of central potentials. The Coulomb potential has been considered in [7] in the context of a two-dimensional potential as generated by the cosmic string itself.

Here we consider a general radial problem and define the spherical harmonics taking into account the angular deficit. We apply the results to the (3 + 1)D Coulomb and harmonic oscillator problems obtaining the spectra for these potentials. These spectra have been independently obtained in [9]. We analyse the harmonic oscillator in (2 + 1)D space subjected to an angular deficit by an algebraic procedure and introduce a new set of ladder operators that are obtained by taking non-integer powers of the usual ones. We consider also the generalization of these two quantum mechanics problems to an $(N+1)$-dimensional spacetime with conic topological structure. These can be originated by an $(N-2)$-brane of cosmic character. The central potential is added with the origin attached to a point of the brane. The well-known relationship between oscillator and Coulomb problems [8] is then generalized to hyper-conic spacetimes.

The structure of the paper is as follows. In section 2 the solution of the potential problems in (3 + 1) dimensions is addressed. The spherical harmonics with angular deficit are constructed. The complete eigenfunctions for the Coulomb and oscillator problems are presented and the ladder operators for the latter potential introduced. The hidden rotational symmetry is discussed. In section 3 the $(N+1)$-dimensional generalization is performed. The dependence of the quadratic angular momentum Casimir operator eigenvalue on the angular deficit is obtained. The spectra of the Coulomb and oscillator problems are obtained and related. Section 4 presents the final comments.

2. Coulomb and quantum oscillator problems in (3 + 1)-D conical spacetime

Let us consider spherical coordinates (r, θ, ϕ) in which the cosmic string metric tensor reads $g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -r^2\sin^2\theta)$ and the Schrödinger equation is written in standard form

$$\left[\nabla_r^2 - \frac{2\mu}{\hbar^2}V(r) + \frac{2\mu E}{\hbar^2}\right]\psi(\boldsymbol{r}) - \frac{\boldsymbol{L}^2}{\hbar^2 r^2}\psi(\boldsymbol{r}) = 0 \tag{3}$$

where

$$\boldsymbol{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta} \right) + \frac{1}{\sin^2\theta} \frac{\mathrm{d}^2}{\mathrm{d}\Phi^2} \right]$$

is the angular momentum operator and $V(r)$ is a central potential whose origin is coincident with a point of the string.

2.1. Eigenfunctions and energy spectrum

Let us perform the separation of variables expressing

$$\psi(\boldsymbol{r}) = R(r)Y(\theta,\phi). \tag{4}$$

Substituting in equation (3) we obtain the set of equations

$$\left[r\frac{\mathrm{d}^2}{\mathrm{d}r^2}r - \frac{2\mu}{\hbar^2}V(r) + \frac{2\mu E}{\hbar^2}r^2 \right] R(r) = \ell(\ell+1)R(r) \quad \text{and} \quad \boldsymbol{L}^2 Y(\theta,\phi) = \hbar^2\ell(\ell+1)Y(\theta,\phi) \tag{5}$$

where $\ell(\ell+1)$ is to be specified later. Separating the second equation with $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$ leads to

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\Theta(\theta) = \lambda^2\Theta(\theta) \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Phi(\phi) = -\lambda^2\Phi(\phi) \tag{6}$$

where λ is another parameter to be determined.

2.1.1. Spherical harmonics. The presence of the string is displayed in the azimuthal equation which is subject to the periodic boundary conditions (2). So, we have

$$\Phi_\alpha^m(\phi) = \frac{1}{\sqrt{2\pi\alpha}}\,\mathrm{e}^{\mathrm{i}\frac{m}{\alpha}\phi} \tag{7}$$

where $m = 0, \pm 1, \pm 2, \ldots, \alpha \in \mathrm{Re}^+$ and we identified $\lambda = \frac{|m|}{\alpha}$. Requiring regular solutions in $\theta = 0, \pi$ we obtain $\ell = k + \frac{|m|}{\alpha}$ and the polar solutions

$$\Theta_k^{\frac{|m|}{\alpha}}(\theta) = \sqrt{\frac{\left(2k + 2\frac{|m|}{\alpha} + 1\right)\Gamma(k+1)}{2\Gamma\left(k + 2\frac{|m|}{\alpha} + 1\right)}}\,\sin^{\frac{|m|}{\alpha}}\theta\, T_k^{\frac{|m|}{\alpha}}(u) \tag{8}$$

where $k = 0, 1, 2, \ldots, u = \cos\theta$ and $T_k^{\frac{|m|}{\alpha}}(u)$ are the Gegenbauer polynomials. Therefore, the generalization of the spherical harmonics required by the cosmic string spacetime is

$$Y_\ell^{\frac{m}{\alpha}}(\theta,\phi) = \sqrt{\frac{(2\ell+1)\Gamma\left(\ell - \frac{|m|}{\alpha} + 1\right)}{4\pi\alpha\Gamma\left(\ell + \frac{|m|}{\alpha} + 1\right)}}\,\sin^{\frac{|m|}{\alpha}}\theta\, T_{\ell - \frac{|m|}{\alpha}}^{\frac{|m|}{\alpha}}(\cos\theta)\,\mathrm{e}^{\mathrm{i}\frac{m}{\alpha}\phi}. \tag{9}$$

For the eigenvalues of the quadratic Casimir operator we observe a dependence on two integers and on the angular deficit α which turns the eigenvalues $\ell(\ell+1)$ to non-integer numbers:

$$\ell(\ell+1) = \left(k + \frac{|m|}{\alpha}\right)\left(k + \frac{|m|}{\alpha} + 1\right). \tag{10}$$

This dependence of the ℓ value on each of the states within a specific angular momentum multiplet can be understood since the operators $\boldsymbol{L}_\pm = \boldsymbol{L}_x \pm \mathrm{i}\boldsymbol{L}_y$ are not operators that act within the Hilbert space. They would create states with the wrong periodicity conditions. The algebraic construction of the angular momentum states is spoiled from the beginning. We will argue later that an attempt to redefine these operators to act in the Hilbert space as $(\boldsymbol{L}_\pm)^{\frac{1}{\alpha}}$ will not work.

2.1.2. Radial equations. The above procedure is valid for any problem with a radial potential $V(r)$. Now we choose a particular potential to solve the radial equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{2\mu}{\hbar^2}V(r) + \frac{2\mu E}{\hbar^2}\right]R(r) = \left(k + \frac{|m|}{\alpha}\right)\left(k + \frac{|m|}{\alpha} + 1\right)R(r). \tag{11}$$

Coulomb potential. The Coulomb potential is expressed by $\frac{-e^2}{r}$. Substituting this in equation (13) we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{2}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{\ell(\ell+1)}{\rho^2} + \frac{\beta}{\rho} - \frac{1}{4}\right]R(\rho) = 0 \tag{12}$$

where β is an arbitrary parameter, $\rho = \frac{r}{\beta r_0}$, $\ell = k + \frac{|m|}{\alpha}$, $E = -\frac{\epsilon_0}{\beta^2}$; with $r_0 = \frac{\hbar^2}{2\mu e^2}$ (Bohr radius divided by two) and $\epsilon_0 = \frac{\mu e^4}{2\hbar^2}$ (ionization energy). With the ansatz

$$R(\rho) = C\,\mathrm{e}^{(-\frac{\rho}{2})}\rho^{\ell}g(\rho) \tag{13}$$

we are led to the radial equation for $g(\rho)$

$$\rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}g(\rho) + (2\ell + 2 - \rho)\frac{\mathrm{d}}{\mathrm{d}\rho}g(\rho) + (\beta - \ell - 1)g(\rho) = 0. \tag{14}$$

This is the associated Laguerre equation whose normalized solutions are the associated Laguerre polynomials

$$L_j^{2\ell+1}(\rho) = \frac{\Gamma(2\ell + j + 2)}{\Gamma(j+1)}\frac{\mathrm{e}^{\rho}}{\rho^{2\ell+1}}\frac{\mathrm{d}^j}{\mathrm{d}\rho^j}[\rho^{2\ell+j+1}\,\mathrm{e}^{-\rho}] \qquad \text{with} \quad j = 0, 1, 2 \ldots.$$

Therefore, the solution for the radial differential equation is

$$R(r) = C^{\alpha}_{j,k,m}\left(\frac{1}{\beta}\frac{r}{r_0}\right)^{k+\frac{|m|}{\alpha}}\mathrm{e}^{-\frac{1}{2\beta}\frac{r}{r_0}}L_j^{2\left(k+\frac{|m|}{\alpha}\right)+1}\left(\frac{1}{\beta}\frac{r}{r_0}\right) \tag{15}$$

where β is given by $j + k + \frac{|m|}{\alpha}$ and $C^{\alpha}_{j,n,m}$ is a normalization constant obtained as

$$C^{\alpha}_{j,k,m} = \frac{1}{\left(j + k + \frac{|m|}{\alpha}\right)}\sqrt{\frac{\Gamma(j)}{2r_0^3\left(\Gamma\left(j + 2k + 2\frac{|m|}{\alpha} + 1\right)\right)^3}}. \tag{16}$$

For the energy spectrum we obtain

$$E^{\alpha}_{j,k,m} = -\frac{\epsilon_0}{\left(j + k + \frac{|m|}{\alpha}\right)^2}. \tag{17}$$

Clearly, the essential degeneracy is broken, but there is still an accidental degeneracy associated with a full symmetry of the potential [10]. It is important to point out that the energy levels depend explicitly on the angular deficit α which characterizes the global structure of the metric.

Quantum harmonic oscillator. Proceeding as done before with the hydrogen atom, we can solve the radial equation (12) for $V(r) = \frac{1}{2}\mu\omega^2 r^2$ given by

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mu^2}{\hbar^2}\omega^2 r^2 + \frac{2\mu E}{\hbar^2}\right]R(r) = \left(k + \frac{|m|}{\alpha}\right)\left(k + \frac{|m|}{\alpha} + 1\right)R(r). \tag{18}$$

Making the ansatz

$$R(x) = Cx^{\ell}\,\mathrm{e}^{-\frac{1}{2}x^2}h(x) \tag{19}$$

where $x = \left(\frac{r}{r_0}\right)^2$ and $r_0 = \sqrt{\frac{\hbar}{\mu\omega}}$. And using equation (19) in (18) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}h(x) + \left(\frac{2\ell+2}{x} - 2x\right)\frac{\mathrm{d}}{\mathrm{d}x}h(x) + \left(\frac{2E}{\hbar\omega} - 2\ell - 3\right)h(x) = 0. \tag{20}$$

Making a new change in variable $x \to x' = x^{\frac{1}{2}}$, we obtain the same equation as in the hydrogen atom (14) with solution $h\left(x^{\frac{1}{2}}\right) = L_j^{\ell+\frac{1}{2}}(x)$. Therefore, the solution for the radial function $R(r)$ is given by

$$R(r) = C_{j,k,m}^{\alpha}\left(\frac{r}{r_0}\right)^{k+\frac{|m|}{\alpha}} \mathrm{e}^{-\frac{1}{2}\left(\frac{r}{r_0}\right)^2} L_j^{k+\frac{|m|}{\alpha}+\frac{1}{2}}\left(\frac{r^2}{r_0^2}\right) \tag{21}$$

and the normalization constant is

$$C_{j,k,m}^{\alpha} = \sqrt{\frac{2}{r_0^3}\frac{\Gamma(j+1)}{\left[\Gamma\left(j+k+\frac{|m|}{\alpha}+\frac{3}{2}\right)\right]^3}}. \tag{22}$$

For the energy spectrum we obtain

$$E_{j,k,m}^{\alpha} = \hbar\omega\left(2j + k + \frac{|m|}{\alpha} + \frac{3}{2}\right). \tag{23}$$

Again, there is a dependence on deficit angle and the degeneracy attributed to symmetry rotations (essential degeneracy) is broken. But we see clearly a persistence of the accidental degeneracy related with even k states for $m = 0$.

Creation and destruction operators. The last section results show that the eigenvalues for the harmonic oscillator increase in intervals of $\hbar\omega$ or of $\hbar/\omega\alpha$. This suggests to investigate the construction of ladder operators for the harmonic oscillator which shall produce these changes. Since the Hilbert space can be constructed as the tensor product $\mathcal{E}_{\rho,\phi} \otimes \mathcal{E}_z$, it is sufficient to consider the 2D quantum harmonic oscillator with angular deficit. The ladder operators acting within $\mathcal{E}_z$ are the usual ones.

The eigenfunctions of the two-dimensional harmonic oscillator, $V(\rho) = \frac{1}{2}\mu\omega^2\rho^2$, with angular deficit, obtained by solving the Schrödinger equation under the boundary conditions given by equation (2) are

$$\Psi_{n,m}(\rho,\phi) = c_{n,m}\rho^{\frac{|m|}{\alpha}} \mathrm{e}^{-\frac{1}{2}\frac{\mu\omega}{\hbar}\rho^2 + \mathrm{i}\frac{m}{\alpha}\phi} L_n^{\frac{|m|}{\alpha}}\left(\frac{\mu\omega}{\hbar}\rho^2\right). \tag{24}$$

Here $n = 0, 1, 2, \ldots$, and $m = 0, \pm 1, \pm 2, \ldots$. The associated energies are

$$E_{n,m} = \hbar\omega\left(2n + \frac{|m|}{\alpha} + 1\right). \tag{25}$$

The Hamiltonian for the 2D quantum harmonic oscillator without angular deficit can be written in terms of creation and destruction operators of right and left 'circular quantum'

$$\boldsymbol{H} = \hbar\omega(\boldsymbol{a}_d^{\dagger}\boldsymbol{a}_d + \boldsymbol{a}_g^{\dagger}\boldsymbol{a}_g + 1). \tag{26}$$

These operators are defined in terms of the usual $\boldsymbol{a}_x$ and $\boldsymbol{a}_y$ operators by

$$\boldsymbol{a}_d = \frac{1}{\sqrt{2}}(\boldsymbol{a}_x - \mathrm{i}\boldsymbol{a}_y) \qquad \text{and} \qquad \boldsymbol{a}_g = \frac{1}{\sqrt{2}}(\boldsymbol{a}_x + \mathrm{i}\boldsymbol{a}_y) \tag{27}$$

the nonzero commutation relations between the four operators $\boldsymbol{a}_d$, $\boldsymbol{a}_g$, $\boldsymbol{a}_d^{\dagger}$ and $\boldsymbol{a}_g^{\dagger}$ being

$$[\boldsymbol{a}_d, \boldsymbol{a}_d^{\dagger}] = [\boldsymbol{a}_g, \boldsymbol{a}_g^{\dagger}] = 1. \tag{28}$$

These relations lead to

$$\left[\boldsymbol{H}, \left(\boldsymbol{a}_{d(g)}\right)^n\right] = -n\left(\boldsymbol{a}_{d(g)}\right)^n \hbar\omega \qquad \text{and} \qquad \left[\boldsymbol{H}, \left(\boldsymbol{a}^{\dagger}_{d(g)}\right)^n\right] = n\left(\boldsymbol{a}^{\dagger}_{d(g)}\right)^n \hbar\omega. \tag{29}$$

In the case of angular deficits neither the $\boldsymbol{a}_{x(y)}$ nor the $\boldsymbol{a}_{d(g)}$ operators can be defined as operators acting on the Hilbert space. The reason is that they do not respect the periodicity of space, leading from states that respect the periodicity, belonging to the Hilbert space, to states that do not respect, outside the Hilbert space. The circular operators are more suited for the discussion of the deficit angular case as we are going to show. The products, $\boldsymbol{a}^{\dagger}_d\boldsymbol{a}_d$ and $\boldsymbol{a}^{\dagger}_g\boldsymbol{a}_g$, that appear in (26) are well-defined operators acting on the Hilbert space since these products do not change the periodicity in angular variables. The decomposition of the Hamiltonian in equation (26) is allowed in the conic spacetime. For this it is useful to consider the space of functions that represents the Hilbert space as embedded in a larger space of (multi-valued) functions of arbitrary periodicity. Then the operator $\boldsymbol{a}_{d(g)}$ acting in a function with the periodicity required for it to represent a state of the Hilbert space changes its periodicity. In the conic space it will correspond to a multi-valued function. This function will not be associated with a state in the Hilbert space. Nevertheless acting now with $\boldsymbol{a}^{\dagger}_{d(g)}$ over that state restores the periodicity needed for it to correspond to a state belonging to the Hilbert space. It is seen then that the Hamiltonian turns out to be a well-defined operator acting within the Hilbert space, as it should be. Let us now discuss the construction of creation and annihilation operators that are allowed to act, individually, on the Hilbert space. In order to properly define the ladder operators acting within the Hilbert space it is necessary to define fractionary powers of the usual creation and annihilation operators. Since the creation and annihilation operators are non-Hermitian operators, their (fractionary) power should not be defined through their spectral decomposition. One particularly simple way to define these highly non-local operators is using the infinite series

$$\left(\boldsymbol{a}^{\dagger}_{d(g)}\right)^{\frac{1}{\alpha}} = \lim_{\epsilon\to 0}\left(\epsilon + \boldsymbol{a}^{\dagger}_{d(g)}\right)^{\frac{1}{\alpha}} = \lim_{\epsilon\to 0}\sum_{q=0}^{\infty} C_{q,\frac{1}{\alpha},\epsilon}\left(\boldsymbol{a}^{\dagger}_{d(g)}\right)^q \tag{30}$$

where the regulating parameter ϵ is to be removed after summing the series and $C_{q,\beta,\epsilon} = \epsilon^{\beta-q}\beta!/((\beta-q)!q!)$. In this way it is straightforward to extend the commutation relations (29), by computing the commutation relations term by term

$$\begin{aligned}\left[\boldsymbol{H}, \left(\boldsymbol{a}_{d(g)}\right)^{\frac{n}{\alpha}}\right] &= \lim_{\epsilon\to 0}\sum_{q=0}^{\infty} C_{q,\frac{n}{\alpha},\epsilon}\left[\boldsymbol{H}, \left(\boldsymbol{a}_{d(g)}\right)^q\right] \\ &= \lim_{\epsilon\to 0}\sum_{q=1}^{\infty} C_{q,\frac{n}{\alpha},\epsilon}(-q\hbar\omega)\left(\boldsymbol{a}_{d(g)}\right)^q \\ &= (-\hbar\omega)\lim_{\epsilon\to 0}\sum_{q=0}^{\infty}\frac{n}{\alpha}C_{q,\frac{n-\alpha}{\alpha},\epsilon}\left(\boldsymbol{a}_{d(g)}\right)^{q+1} \\ &= -\hbar\omega\frac{n}{\alpha}\lim_{\epsilon\to 0}\left(\epsilon + \boldsymbol{a}_{d(g)}\right)^{(\frac{n}{\alpha}-1)}\boldsymbol{a}_{d(g)}.\end{aligned} \tag{31}$$

Repeating the argument with creation operators we obtain the result

$$\begin{aligned}\left[\boldsymbol{H}, \left(\boldsymbol{a}_{d(g)}\right)^{\frac{n}{\alpha}}\right] &= -\frac{n}{\alpha}\left(\boldsymbol{a}_{d(g)}\right)^{\frac{n}{\alpha}}\hbar\omega \\ \left[\boldsymbol{H}, \left(\boldsymbol{a}^{\dagger}_{d(g)}\right)^{\frac{n}{\alpha}}\right] &= \frac{n}{\alpha}\left(\boldsymbol{a}^{\dagger}_{d(g)}\right)^{\frac{n}{\alpha}}\hbar\omega\end{aligned} \tag{32}$$

which allows for an interpretation in terms of fractionary quantum creation and destruction operators.

It is also straightforward to obtain the action of fractionary power operators on the ground state

$$
\begin{aligned}
\langle\rho,\phi|\big(\boldsymbol{a}^{\dagger}_{g(d)}\big)^{\frac{n'}{\alpha}}|0,0\rangle &= \lim_{\epsilon\to 0}\sum_{q=0}^{\infty} C_{q,\frac{n'}{\alpha},\epsilon}\langle\rho,\phi|\big(\boldsymbol{a}^{\dagger}_{g(d)}\big)^{q}|0,0\rangle \\
&= \lim_{\epsilon\to 0}\sum_{q=0}^{\infty} C_{q,\frac{n'}{\alpha},\epsilon}\sqrt{\frac{\mu\omega}{\pi\hbar}}\left(\frac{\mu\omega}{\hbar}\rho\right)^{q}\mathrm{e}^{-\mathrm{i}q\phi-\frac{\mu\omega}{2\hbar}\rho^{2}} \\
&= \lim_{\epsilon\to 0}\sqrt{\frac{\mu\omega}{\pi\hbar}}\mathrm{e}^{-\frac{\mu\omega}{2\hbar}\rho^{2}}\left(\epsilon+\frac{\mu\omega}{\hbar}\rho\mathrm{e}^{-\mathrm{i}\phi}\right)^{\frac{n'}{\alpha}} \\
&= \sqrt{\frac{\mu\omega}{\pi\hbar}}\mathrm{e}^{-\frac{\mu\omega}{2\hbar}\rho^{2}-\mathrm{i}\frac{n'}{\alpha}\phi}\left(\frac{\mu\omega}{\hbar}\rho\right)^{\frac{n'}{\alpha}}.
\end{aligned}
\tag{33}
$$

This gives the eigenfunctions obtained by the direct solution of the differential equations. Note that each term of this infinite series does not represent a state belonging to the Hilbert space. Nevertheless the function obtained after summing the series and removing thereafter the regulating parameter ϵ has the required periodicity for it to represent a state belonging to the Hilbert space.

The operator product $(\boldsymbol{a}^{\dagger}_{g}\boldsymbol{a}^{\dagger}_{d})$ also does not change the periodicity condition and can be in principle defined in the Hilbert space. Indeed the axially symmetric states not depending on the angular variable are eigenstates in both the usual and in the conic space cases, being insensitive to the topological defect of the spacetime. These states are created by applying this operator product on the fundamental state. Also the action of this product of ordinary operators on states with angular dependence, that were constructed from the action of fractionary operators on the fundamental states, is allowed and does not affect its angular dependence. We are thus led to construct the general basis state vector, corresponding to eigenfunctions (24), as

$$
|n,n'\rangle^{\alpha}_{g(d)} = (\boldsymbol{a}^{\dagger}_{g}\boldsymbol{a}^{\dagger}_{d})^{n}\big(\boldsymbol{a}^{\dagger}_{g(d)}\big)^{\frac{n'}{\alpha}}|0,0\rangle. \tag{34}
$$

Note that this exhausts all states corresponding to equation (24). Note also that in equation (34) we avoided the use of the fractionary power operators $(\boldsymbol{a}^{\dagger}_{g})^{\frac{1}{\alpha}}$ and $(\boldsymbol{a}^{\dagger}_{d})^{\frac{1}{\alpha}}$ at the same time. Indeed inspection shows that the action of their product on the fundamental state, although not forbidden by periodicity considerations, leads to non-normalizable states. Therefore, all states of the model are given by linear combinations of

$$
|n,n'\rangle^{\alpha}_{g} \qquad \text{and} \qquad |n,n'\rangle^{\alpha}_{d}. \tag{35}
$$

The energy of the basis states can be calculated, through relations (29) and (32), as

$$
E = \hbar\omega\left(2n+\frac{n'}{\alpha}+1\right). \tag{36}
$$

Let us now use these operators to discuss some questions related to the hidden symmetry of the oscillator. It is well known that the angular momentum algebra describes the degeneracy of the 2D oscillator. The 'angular momentum' operators are defined as

$$
\boldsymbol{J}_{\pm} = \boldsymbol{a}^{\dagger}_{d(g)}\boldsymbol{a}_{g(d)} \tag{37}
$$

and

$$
\boldsymbol{J}_{z} = \tfrac{1}{2}(\boldsymbol{a}^{\dagger}_{d}\boldsymbol{a}_{g} - \boldsymbol{a}^{\dagger}_{g}\boldsymbol{a}_{d}). \tag{38}
$$

The Casimir operator $\boldsymbol{J}^2$ is

$$
\boldsymbol{J}^{2} = \tfrac{1}{2}(\boldsymbol{J}_{+}\boldsymbol{J}_{-}+\boldsymbol{J}_{-}\boldsymbol{J}_{+})+\boldsymbol{J}_{z}^{2} = \tfrac{1}{2}(\boldsymbol{N}_{g}+\boldsymbol{N}_{d})\big[\tfrac{1}{2}(\boldsymbol{N}_{g}+\boldsymbol{N}_{d})+1\big] \tag{39}
$$

where

$$\boldsymbol{N}_g + \boldsymbol{N}_d = \boldsymbol{a}_g^\dagger \boldsymbol{a}_g + \boldsymbol{a}_d^\dagger \boldsymbol{a}_d = \frac{\boldsymbol{H}}{\hbar\omega} - 1. \tag{40}$$

This leads to the identification of the J quantum number associated with the square of the angular momentum with the eigenvalue of $\frac{(\boldsymbol{N}_g+\boldsymbol{N}_d)}{2}$

$$J = \left\langle \frac{N_g + N_d}{2} \right\rangle. \tag{41}$$

In the case of angular deficit neither the $\boldsymbol{J}_+$ nor the $\boldsymbol{J}_-$ operators are defined as operators acting within the Hilbert space of the 2D harmonic oscillator spoiling the hidden $SU(2)$ symmetry. These operators should be exchanged by $\boldsymbol{J}_\pm^{1/\alpha}$ to act in the Hilbert space. Nevertheless the composite operators $\boldsymbol{J}^2$, $\boldsymbol{J}_z$, $\boldsymbol{N}_g$ and $\boldsymbol{N}_d$ are *bona fide* operators. The relationship expressed by equations (39)–(41) is thus extended to the deficit angular space case. Let us consider then the action of $\boldsymbol{N}_g + \boldsymbol{N}_d$ on the basis states of equation (34). Taking $n' = 2m$ in that equation it is straightforward to see that

$$\frac{\boldsymbol{N}_g + \boldsymbol{N}_d}{2} |n, 2m\rangle_{g(d)}^\alpha = \left(n + \frac{|m|}{\alpha}\right) |n, 2m\rangle_{g(d)}^\alpha. \tag{42}$$

In other words these states have quantum numbers

$$j = n + \frac{|m|}{\alpha}. \tag{43}$$

This reproduces exactly the form obtained in section 2.1 by the resolution of the angular differential equations in (3 + 1)-D spacetime.

It can also be understood why the operators $(J_\pm)^{1/\alpha}$ do not generate all states of a multiplet. Since $a_{g(d)}^\dagger$ and $a_{g(d)}$ appear simultaneously with fractionary powers in $(J_\pm)^{1/\alpha}$ they generate non-normalizable functions when applied to the basis states of equation (34). This is necessary to allow for the change in j value within the multiplet.

3. (*N* + 1)-dimensional generalization

In order to construct an N-dimensional generalization for Coulomb and quantum oscillator problems we consider the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(\mathrm{d}\rho^2 + \frac{1}{\rho^2}\,\mathrm{d}\phi^2 + \mathrm{d}x_3^2 + \cdots + \mathrm{d}x_N^2\right). \tag{44}$$

The variable ϕ is assumed to present an angular deficit, $\phi \to \phi\alpha$. This spacetime generalizes the cosmic string spacetime and the $(N-2)$-brane is considered in $x_1 = x_2 = 0$. We have taken out one generator of the angular momentum algebra and assumed non-trivial boundary conditions for the orbit it generates in real space breaking thus the $SO(N)$ symmetry.

We introduce the hyper-conical spacetime using hyper-spherical coordinates according to [11]. The metrics reads

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -r^2\sin^2\theta, -r^2\sin^2\theta\sin^2\phi_1, \ldots, -r^2\sin^2\theta\sin^2\phi_1\cdots\sin^2\phi_{N-1}) \tag{45}$$

where

$$\begin{aligned} &0 \leqslant r \leqslant \infty \\ &0 \leqslant \theta \leqslant \pi \\ &-\pi \leqslant \phi_1 \leqslant \pi \\ &\vdots \\ &-\pi\alpha \leqslant \phi_{N-1} \leqslant \pi\alpha. \end{aligned} \tag{46}$$

The modified range of the variable ϕ_{N-1} is responsible by the conical character. In this way the N-dimensional Schrödinger equation can be written as [12]

$$\left[\nabla_r^2 - \frac{\boldsymbol{L}^2}{\hbar r^2} - \frac{\mu V(r)}{\hbar^2} + \frac{2\mu E}{\hbar^2}\right]\Psi(r) = 0 \tag{47}$$

where

$$\nabla_r^2 = \frac{1}{r^{N-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1}\frac{\mathrm{d}}{\mathrm{d}r}\right) \tag{48}$$

and

$$\begin{aligned} \boldsymbol{L}^2 = {} & \frac{1}{\sin^{N-2}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^{N-2}\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + \frac{1}{\sin^2\theta\sin^{N-3}\phi_1}\frac{\mathrm{d}}{\mathrm{d}\phi_1}\left(\sin^{N-3}\phi_1\frac{\mathrm{d}}{\mathrm{d}\phi_1}\right) \\ & + \cdots + \frac{1}{\sin^2\theta\sin^2\phi_1\cdots\sin^2\phi_{N-4}}\frac{1}{\sin\phi_{N-3}}\frac{\mathrm{d}}{\mathrm{d}\phi_{N-3}}\left(\sin\phi_{N-3}\frac{\mathrm{d}}{\mathrm{d}\phi_{N-3}}\right) \\ & + \frac{1}{\sin^2\theta\sin^2\phi_1\cdots\sin^2\phi_{N-3}}\frac{\mathrm{d}^2}{\mathrm{d}\phi_{N-2}^2}. \end{aligned} \tag{49}$$

For $V(r)$ to be strictly radial we can perform the separation of variables method N times and obtain the angular equation

$$\boldsymbol{L}^2 Y_{n_0,\ldots,n_{N-3}}(\theta,\phi_1,\ldots,\phi_{N-2}) = \ell(\ell+N-2)\hbar^2 Y_{n_0,\ldots,n_{N-3}}(\theta,\phi_1,\ldots,\phi_{N-2}) \tag{50}$$

where N is the number of spatial dimensions. Introducing

$$k = \sum_{i=0}^{N-3} n_i$$

where the integers n_i are separation constants, the non-trivial boundary condition will affect the quantum number ℓ in the same form as in the three-dimensional case

$$\ell = k + \frac{|m|}{\alpha}. \tag{51}$$

The radial equation will be

$$\left[r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} + r(N-1)\frac{\mathrm{d}}{\mathrm{d}r} + r^2\left(\frac{2\mu E}{\hbar^2} - \frac{\mu}{\hbar^2}V(r)\right)\right]R(r) = \ell(\ell+N-2)R(r). \tag{52}$$

3.1. $(N+1)$-dimensional solutions for the hydrogen atom and quantum harmonic oscillator

In the Coulomb problem the radial solutions turn out to be

$$R(r) = C\left(\frac{r}{r_0}\right)^{\ell} \mathrm{e}^{-\frac{1}{2}\left(\frac{r}{r_0}\right)} L_i^{2\ell+N-2}\left(\frac{r}{kr_0}\right) \tag{53}$$

where $k^2 = -\frac{\epsilon_0}{E}$, $i = 1, 2, \ldots$, and C is a normalization constant. The energy spectrum is

$$E = -\frac{\epsilon_0}{\left(i + \ell + \frac{N-3}{2}\right)^2}. \tag{54}$$

For the oscillator problem the radial solutions are

$$R(r) = C\left(\frac{r}{r_0}\right)^{\ell} \mathrm{e}^{-\frac{1}{2}(\frac{r}{r_0})^2} L_i^{\ell+\frac{N-2}{2}}\left(\frac{r^2}{r_0^2}\right) \tag{55}$$

where C is a normalization constant and $i = 1, 2, \ldots$. The energy spectrum is

$$E = \hbar\omega\left[\ell + 2i + \frac{N-4}{2}\right]. \tag{56}$$

Let us now discuss the relationship between both potential solutions along the lines discussed for trivial topology in [8].

3.2. Relationship

With the generalization of the spacetime above it is straightforward to generalize the mapping [8] of the states of both problems:

	Hydrogen atom	Harmonic oscillator
Radial variable	$\frac{1}{\beta}\frac{r}{r_0}$	$\left(\frac{r}{r_0'}\right)^2$
Energy	$\left(\frac{\epsilon_0}{E}\right)^{\frac{1}{2}}$	$\frac{E'}{2\hbar\omega}$
Generalized angular momentum quantum number	2ℓ	$\ell' + \lambda$
Spatial dimension	N	$\frac{N'}{2} - \lambda + 1$
Azimuthal quantum number	$2\frac{\lvert m\rvert}{\alpha}$	$\frac{\lvert m'\rvert}{\alpha'}$
Angular deficit	α	$\alpha'(2\alpha')$

where λ allows for a freedom in the mapping. For instance if $\lambda = 0$ we have $N' = 2(N-1)$ as discussed in [13] in the case without angular deficit whereas the general case $N \neq 0$ is treated in the usual spacetime in [8].

This mapping reveals that the direct relation of the even states of the quantum harmonic oscillator and all states of hydrogen atom is attainable in the spacetime with angular deficit. It is to be pointed out that the relation can be established with different angular deficits α and α' and azimuthal quantum numbers just keeping $2|m|/\alpha = |m'|/\alpha'$. Also note that, since l and l' are not integers, a specification of the azimuthal quantum numbers m and m' is in order to obtain a mapping between the orbital quantum numbers l and l'. This is not necessary in the usual space models for which only ℓ and ℓ' are needed.

4. Conclusions

In this work, we studied the solutions of the Schrödinger equation in cosmic string-like spacetimes with a point in the string acting as a source for a radial potential. We performed an extension of the spherical harmonics to the non-trivial spacetime. We verified that the global characteristic of the spacetime is present explicitly in the structure of states and energy spectrum. The deficit angle splits the degeneracy associated with rotational symmetry in energy spectrum, but the accidental degeneracy is still partially present.

The extension of the algebraic method of construction of the harmonic oscillator states through the introduction of fractionally powered ladder operators allowed the discussion of its hidden symmetry. By the use of the Schwinger construction of the angular momentum operators some light was shed on the dependence of the angular momentum Casimir operator values on the angular deficit and on the algebraic construction of the angular momentum states. Further developments in this analysis can be foreseen by noting that in terms of the fractionary power creation and destruction operators that act within the Hilbert space and the ordinary number operators $\boldsymbol{N}_{d(g)} = \boldsymbol{a}^{\dagger}_{d(g)}\boldsymbol{a}_{d(g)}$ that also act within the Hilbert space, the Wigner–Heisenberg algebra becomes a kind of deformed algebra. For instance the basic relation should be pointed out, $\boldsymbol{N}_{d(g)}\boldsymbol{a}^{1/\alpha}_{d(g)} = \boldsymbol{a}^{1/\alpha}_{d(g)}(\boldsymbol{N}_{d(g)} - \mathbf{1}/\alpha)$. The fractionary power ladder operators are here defined through the non-perturbative use of infinite series. Although this construction has allowed a consistent treatment of these operators, alternative treatments should be the object of further investigations with the aim of obtaining a greater mathematical rigour as well as of finding less laborious procedures. Strategies based upon the consideration of the power variable as a complex parameter within an analytical extension procedure might be considered. For particular α values the operators introduced here become local. For instance, if $\alpha = 1/2$ the raising operators become $(a^{\dagger})^2$ and the obstruction pointed out to the simultaneous use of left and right raising operators disappears. In these cases the presence of the string affects the quantum states simply as a superselection rule.

Moreover, the remarkable point raised by this study is that the attempt to allow for the relationship between the Coulomb potential in 3D and the oscillator problem in higher dimensions, if the 3D space presents conical topology, leads naturally to the construction of the conic spacetimes of higher dimensions. In this sense a quantum mechanical issue is used as a guide to relate topological spacetimes in different dimensions.

The somewhat artificial consideration of the source of the potential point exactly over the string restricts severely any attempt to use these results as a means of detecting a real cosmic string.

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